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Delay-Independent Stability of a Special Sequence of Neutral Difference–Differential Equations with One Delay

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The equations of the title for the state $X\colon [-T, \infty) \to \mathbf{R}$

$$L[X] := [(D-a)(D-\bar{a})]^n X(t) + B \cdot [(D-b)(D-\bar{b})]^n X(t-T) = 0, \qquad D := d/dt, \tag{1}$$

are considered in the parameter space $(a, b, B, n) \in \mathbf{P}$ where $\mathbf{P}$ is defined by

$$\mathrm{Re}(a) < 0,\ \mathrm{Im}(a) \geqslant 0,\ \mathrm{Im}(b) \geqslant 0,\ B \in \mathbf{R},\ n \in \mathbf{N}.$$

If $a \in \mathbf{R}$, then $\bar{a}$ is a second real point $\bar{a} \neq a$. For $a \in \mathbf{R}$, $\bar{a}$ is the complex conjugate to $a \in \mathbf{C}$. The same notation applies for b. By a tedious but elementary analysis, the set $\mathbf{S}_0 \subset \mathbf{P}$ for which (1) has for all $T \geqslant 0$ only asymptotically stable solutions in the sense of Liapunov is explicitly determined in the form

$$|B| < M^n(a, b) \leqslant 1. \tag{2}$$

The stability boundary decays exponentially with respect to the multiplicity, n, of the zeros of the coefficient polynomials of (1). We generalize a theorem of F. Brauer [*J. Differential Equations* **69** (1987), 185–191] dealing with delay-independent stability for characteristic equations of the form

$$H(z) := A(z) + B(z)\exp(-zT), \qquad T \geqslant 0, \tag{3}$$

where $A(z)$, $B(z)$ are holomorphic in $\mathrm{Re}(z) \geqslant 0$ in such a way that the neutral case can be dealt with.

1. Problem and Motivation

Consider the sequence of linear autonomous neutral difference–differential equations with constant coefficients and one constant delay $T \geqslant 0$

$$L[X] := [(D-a)(D-\bar{a})]^n X(t) + B \cdot [(D-b)(D-\bar{b})]^n \times X(t-T) = u(t), \qquad D := d/dt,\ n \in \mathbf{N}, \tag{1.1}$$

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for the scalar state X: $[-T, +\infty) \to \mathbf{R}$ when u: $\mathbf{R}_+ \to \mathbf{R}$ is given and $X(t)$ is known in $[-T, 0]$. In order to study the asymptotic stability in the sense of Liapunov of the solution X of (1.1) it suffices to study the same stability for the trivial solution of $L[X]=0$ only. The function $X(t) := \exp(zt)$, $z \in \mathbf{C}$, solves $L[X]=0$ if and only if z is a zero of the characteristic function $H(z)$ for (1.1)

$$H(z) := [(z-a)(z-\bar{a})]^n + B \cdot [(z-b)(z-\bar{b})]^n \exp(-zT),$$
$$a \neq b,\ \mathrm{Re}(a) < 0,\ \mathrm{Im}(a) \geqslant 0,\ \mathrm{Im}(b) \geqslant 0. \tag{1.2}$$

We consider (1.1) in the parameter space

$$(a, b, B, n) \in \mathbf{Q}_2 \times \mathbf{I}_+ \times \mathbf{R} \times \mathbf{N} =: \mathbf{P}, \tag{1.3}$$

where $\mathbf{Q}_2 := \{z \in \mathbf{C}\colon \mathrm{Re}(z) < 0, \mathrm{Im}(z) \geqslant 0\}$ is the second quadrant of $\mathbf{C}$ open along the imaginary axis and $\mathbf{I}_+ := \{z \in \mathbf{C}\colon \mathrm{Im}(z) \geqslant 0\}$ is the closed upper half-plane of $\mathbf{C}$. Denote by

$$\mathbf{S}_0 := \{(a, b, B, n) \in \mathbf{P}\colon H(z) = 0 \to \mathrm{Re}(z) < 0 \text{ for all } T \geqslant 0\} \tag{1.4}$$

the delay-independent stable manifold of $H(z)$. Clearly, $(a, b, B, n) \in \mathbf{S}_0$ is necessary for the asymptotic stability of the trivial solution of $L[X]=0$ for all $T \geqslant 0$. In a supplementary discussion in Section 3, we will see under what circumstances the necessary condition is also sufficient. In Section 2, $\mathbf{S}_0$ is explicitly determined for the $\mathbf{P}$ under consideration. Section 3 is devoted to a discussion. Among others, we consider the more general

$$H(z) := A(z) + B(z) \exp(-zT), \tag{1.5}$$

where $A(z)$, $B(z)$ are holomorphic in $\mathrm{Re}(z) \geqslant 0$. The kind of stability considered here is also known as "absolute stability" (see El'sgol'ts and Norkin [11]). We prefer, however, our longer but more informative naming. The letter "i" is exclusively reserved for the imaginary unit i, $i^2 = -1$.

2. The Delay-Independent Manifold $\mathbf{S}_0$

In this section $\bar{a}$ is the complex conjugate of $a \in \mathbf{C}$. For technical reasons, we now consider only the generic case by which we mean

$$\{a, \bar{a}\} \cap \{b, \bar{b}\} = \varnothing. \tag{2.1}$$

The treatment of the non-generic cases is postponed to Section 3. We use the notations

$$A_1 := (a^2 + \bar{a}^2)/2,\ B_1 := (b^2 + \bar{b}^2)/2,\ R_a^2 := a^2\bar{a}^2,\ R_b^2 := b^2\bar{b}^2. \tag{2.2}$$

Let $a := ir_a \exp(i\alpha)$, $r_a > 0$, $0 < \alpha \leqslant \pi/2$, and $b := ir_b \exp(i\beta)$, $r_b \geqslant 0$, $-\pi/2 \leqslant \beta \leqslant \pi/2$. Then the range of $A_1 := \mathrm{Re}(a^2)$ as well as $B_1 := \mathrm{Re}(b^2)$ is $\mathbf{R}$. The complex plane $\mathbf{C}$ is referred to Cartesian coordinates, $z := x + iy$. We use $Y := y^2$.

THEOREM 2.1. *For generic $H(z)$ from* (1.2), $\mathbf{S}_0$ *is given by*

$$\mathbf{S}_0 = \{(a, b, B, n) \in \mathbf{P}: |B| < M^n(a, b)\},$$

where $M^2(a, b)$ is defined by

$$M^2(a,b) := \begin{cases} 1, & A_1 \geqslant B_1, \quad R_a \geqslant R_b, \\ m_2^2 & (A_1 < B_1, R_a \geqslant R_b) \\ & \text{or}\,(A_1 \leqslant \min\{B_1, B_1 R_a^2/R_b^2\}, R_a < R_b), \\ R_a^2/R_b^2, & A_1 > B_1 R_a^2/R_b^2, \quad R_a < R_b, \\ m_1^2, & B_1 < A_1 \leqslant B_1 R_a^2/R_b^2, \quad R_a < R_b, \\ R_a^2/R_b^2, & A_1 = B_1 > 0, \quad R_a < R_b, \\ (R_a^2 - A_1^2)/(R_b^2 - A_1^2), & A_1 = B_1 \leqslant 0, \quad R_a < R_b, \end{cases}$$

$$\begin{aligned} m_j^2 &:= (A_1 + Y_j)/(B_1 + Y_j), \qquad j = 1, 2, \\ Y_j &:= Y_0 + (-1)^j \sqrt{(Y_0^2 + \Delta)}, \\ Y_0 &:= (R_a^2 - R_b^2)/(2B_1 - 2A_1), \\ \Delta &:= (B_1 R_a^2 - A_1 R_b^2)/(B_1 - A_1). \end{aligned} \tag{2.3}$$

Proof. From (2.1) follows that $z = b$ is not a zero of $H(z)$ so that we may write $H(z) = 0$ as

$$[(z-a)(z-\bar{a})/\{(z-b)(z-\bar{b})\}]^n = -B \exp(-zT). \tag{2.4}$$

Consider a zero of $H(z)$ in $\mathrm{Re}(z) \geqslant 0$. For such a zero, we infer from (2.4)

$$|(z-a)(z-\bar{a})/\{(z-b)(z-\bar{b})\}| \leqslant |B|^{1/n}. \tag{2.5}$$

If the solution set $\mathbf{L}$ of (2.5) lies entirely in $\mathrm{Re}(z) < 0$, no zero of $H(z)$ in $\mathrm{Re}(z) \geqslant 0$ is possible. $\mathbf{L}$ consists for small $|B|$ of two ovals lying symmetrically about the real axis and contains the points a, $\bar{a}$. For larger $|B|$, $\mathbf{L}$ is simply connected. This kind of connectivity implies that $\mathbf{L}$ lies in $\mathrm{Re}(z) < 0$ if with $z := x + iy$

$$\begin{aligned} M^2 &:= \inf\{F(y): y \in \mathbf{R}\} > |B|^{2/n}, \\ F(y) &:= [y^4 + 2\,\mathrm{Re}(a^2)\, y^2 + |a|^4]/[y^4 + 2\,\mathrm{Re}(b^2)\, y^2 + |b|^4] \end{aligned} \tag{2.6}$$

holds. With the notations of (2.2), F, as a function of $Y := y^2$, reads

$$F(Y) := [Y^2 + 2A_1 Y + R_a^2]/[Y^2 + 2B_1 Y + R_b^2]. \tag{2.7}$$

It remains to determine M^2 explicitly. In order to do this, we discern the cases $F(Y) \geqslant 1$, $F(Y) \leqslant 1$, and, third, the case in which $F(Y)-1$ takes on both signs and we will treat them in that order.

2.1. *The Case* $F(Y) \geqslant 1$

$F(Y) \geqslant 1$ for all $Y \geqslant 0$ takes place if and only if

$$R_b - R_a \leqslant 0 \leqslant A_1 - B_1. \tag{2.8}$$

This means

$$M^2 := 1, \qquad A_1 \geqslant B_1, \qquad R_a \geqslant R_b. \tag{2.9}$$

2.2. *Some Preparations*

The remaining cases need some analysis. $F(Y) \leqslant 1$ for all $Y \geqslant 0$ entails

$$A_1 - B_1 \leqslant 0 \leqslant R_b - R_a. \tag{2.10}$$

The equation $F(Y) = 1$ is only solvable for $Y \geqslant 0$ if

$$(B_1 - A_1)(R_b - R_a) \leqslant 0, \ A_1 \neq B_1 \tag{2.11}$$

holds. If (2.11) is satisfied then there is a unique non-negative solution, Y, of $F(Y) = 1$

$$Y = Y_0 := -(R_b^2 - R_a^2)/(2B_1 - 2A_1). \tag{2.12}$$

From

$$\begin{aligned} F(Y) &= R_a^2 R_b^{-2}[1 + 2(A_1/R_a^2 - B_1/R_b^2)\, Y + \mathbf{O}(Y^2)], & Y &\to 0, \\ &= 1 + 2(A_1 - B_1)/Y + \mathbf{O}(Y^2), & Y &\to +\infty, \end{aligned} \tag{2.13}$$

the geometric meaning of the signs of $A_1 - B_1$ and $A_1/R_a^2 - B_1/R_b^2$ for the graph of $F(Y)$ becomes apparent; $\operatorname{sgn}(A_1/R_a^2 - B_1/R_b^2) = \operatorname{sgn}(F'(0))$ and $\operatorname{sgn}(A_1 - B_1) = \operatorname{sgn}(F(Y_\infty) - 1)$ where Y_∞ is large and positive. Possible local extrema of $F(Y)$, different from the trivial solution $Y = 0$ are located at the solutions Y of

$$F'/F = 2(Y + A_1)/[Y^2 + 2A_1 Y + R_a^2] - 2(Y + B_1)/[Y^2 + 2B_1 Y + R_b^2] = 0. \tag{2.14}$$

For $A_1 = B_1$, we obtain $Y = -A_1$ and M^2 is easily found for $R_a < R_b$ as given in the last two cases of (2.3). For $A_1 \neq B_1$, we write (2.14) as

$$(Y - Y_0)^2 = Y_0^2 + \Delta =: D, \qquad \Delta := R_a^2 R_b^2 (B_1/R_b^2 - A_1/R_a^2)/(B_1 - A_1). \tag{2.15}$$

Elementary calculations yield $D>0$ in $\mathbf{P}$. Solving (2.14) for Y gives for $j=1,2$

$$Y=Y_j:=Y_0+(-1)^j\sqrt{(Y_0^2+\Delta)}. \tag{2.16}$$

Observe that the Y_j lie symmetric about $Y=Y_0$. The non-negativity of the Y_j depends on the signs

$$\begin{aligned}\sigma_0&:=\operatorname{sgn}(Y_0)=-\operatorname{sgn}(B_1-A_1)\operatorname{sgn}(R_b-R_a),\\ \sigma_\Delta&:=\operatorname{sgn}(\Delta)=\operatorname{sgn}(B_1-A_1)\operatorname{sgn}(B_1/R_b^2-A_1/R_a^2).\end{aligned} \tag{2.17}$$

In the parameter domain specified by (2.10) we have

$$\sigma_\Delta\in\{1,\operatorname{sgn}(-A_1)\}. \tag{2.18}$$

For $\sigma_\Delta\leqslant 0$ in the set (2.10), one has to fulfill the system

$$A_1\leqslant B_1,\qquad R_a\leqslant R_b,\qquad B_1\leqslant A_1R_b^2/R_a^2. \tag{2.19}$$

Under (2.8), $\sigma_\Delta=-1,0,1$ is possible. Next, we consider the domain of (2.11). The inequality $\sigma_\Delta\leqslant 0$ is equivalent to the system

$$(B_1-A_1)(R_b-R_a)\leqslant 0,\qquad (B_1-A_1)(B_1-A_1R_b^2/R_a^2)\leqslant 0. \tag{2.20}$$

Again, $\sigma_\Delta=-1$ is possible for both signs of B_1-A_1.

We are in a position to calculate M^2 for the remaining cases.

2.3. *The Case $F(Y)\leqslant 1$*

For $R_a<R_b$ both signs of $F'(0)$ are possible. This, together with $Y_0\leqslant 0$, leads to

$$M^2:=\begin{cases}F(Y_2), & A_1<\min\{B_1,B_1R_a^2/R_b^2\},\quad R_a<R_b,\\ R_a^2/R_b^2, & B_1R_a^2/R_b^2\leqslant A_1<B_1,\quad R_a<R_b.\end{cases} \tag{2.21}$$

2.4. *The Case of Both Signs of $F(Y)-1$*

In the present case $Y_0>0$ combined with the signs of $F(0)-1$, $F'(0)$ results in

$$M^2:=\begin{cases}F(Y_1), & B_1<A_1\leqslant B_1R_a^2/R_b^2,\quad R_a<R_b,\\ R_a^2/R_b^2, & A_1>\max\{B_1,B_1R_a^2/R_b^2\},\quad R_a<R_b,\\ F(Y_2), & A_1<B_1,\quad R_a>R_b.\end{cases} \tag{2.22}$$

Equations (2.9), (2.21), and (2.22) form a complete case distinction. Or, in other words, the considered parameter space $\mathbf{P}$ is covered. Finally, we simplify the expression for $F(Y_j)$. If Y solves $F'/F=0$ then

$$F(Y)=[Y^2+2A_1Y+R_a^2]/[Y^2+2B_1Y+R_b^2]=(Y+A_1)/(Y+B_1)=:m^2(y) \tag{2.23}$$

which explains the expression for $m_j^2 := m^2(y_j)$ in (2.3). Uniting (2.9), (2.21), and (2.22) gives (2.3). So far, we have shown that the set $\mathbf{S}'$ defined in (2.3) belongs to $\mathbf{S}_0$. We have not yet shown that $\mathbf{S}_0$ is not larger than $\mathbf{S}'$. Denote by $\mathbf{C}_T$ the solution curve of

$$|(z-a)(z-\bar{a})/[(z-b)(z-\bar{b})]|^n = |-B\exp(-zT)|. \tag{2.24}$$

All zeros of $H(z)$ lie on $\mathbf{C}_T$. The kth zero, $k \in \mathbf{Z}$, of $H(z)$ lies at the intersection of $\mathbf{C}_T$ with the curve $\mathbf{S}_k$ defined by

$$yT + n\{\arg[(z-a)(z-\bar{a})] - \arg[(z-b)(z-\bar{b})]\} = (2k+\beta)\pi, \tag{2.25}$$

where $\beta := \arg(-B)/\pi$. Consider a point $z_* := iy_*$, $y_* > 0$, and define $|B_*|$ by (2.5) with the equality sign. All curves $\mathbf{C}_T$ of the family $\{\mathbf{C}_T\colon T \geqslant 0\}$ pass through z_*. Solving (2.25) at z_* for T yields $T_* := T(y_*)$ with

$$\begin{aligned} T(y) := \big[(2k+\beta)\,\pi + n[&\arctan\{(|a|^2 - y^2)/(2y\,|\mathrm{Re}(a)|)\} \\ &+ \pi\{1 + \mathrm{sgn}[\mathrm{Re}(b)]\}/2 \\ &+ \mathrm{sgn}[\mathrm{Re}(b)]\arctan\{(|b|^2 - y^2)/(2y\,|\mathrm{Re}(b)|)\}]\big]/y. \end{aligned} \tag{2.26}$$

In (2.6) the principal branch of the arctan-function is to be taken and $k \in \mathbf{N}$ must be sufficiently large ($k \geqslant n$ suffices). This means that whenever $|B| \geqslant M^n(a, b)$ is fulfilled for a finite point on $\mathrm{Re}(z) = 0$, $z \neq 0$, then there is a $T = T_*$ which yields a zero of $H(z)$ on $\mathrm{Re}(z) = 0$. So $\mathbf{S}_0$ cannot be enlarged. This completes the proof. ∎

3. Discussion

For the sake of completeness, we handle the degenerate case for which $H(z)$ factors,

$$H(z) := [(z-a)(z-\bar{a})]^n\,\{1 + B\exp(-zT)\}. \tag{3.1}$$

Evidently, for $H(z)$ from (3.1),

$$\mathbf{S}_0 := \{(a, b, B, n) \in \mathbf{P}\colon |B| < 1\}. \tag{3.2}$$

Note that $\mathrm{Re}(a) < 0$ was incorporated in the definition of $\mathbf{P}$. It is easy to see that delay-independent stability is impossible for $\mathrm{Re}(a) > 0$. Now, consider

$$H(z) = A(z) + B \cdot B(z)\exp(-zT), \tag{3.3}$$

where $A(z)$, $B(z)$ are real monic polynomials of degree $2n$. Restrict $H(z)$ to the subclass in which $A(z)$, $B(z)$ have exactly two different roots. The cases of that class which are not covered by Theorem 2.1 are those in which at least one of the pairs $\{a, \bar{a}\}$, $\{b, \bar{b}\}$ is a pair of real different points. Consider such a real pair, say $\{a, \bar{a}\}$ for definiteness. We obtain for the corresponding $F(Y)$ from (2.6)

$$
\begin{aligned}
F(y) &:= (y^2+a^2)(y^2+\bar{a}^2)/[(y^2+b^2)(y^2+\bar{b}^2)] \\
&= [Y^2+2A_1Y+R_a^2]/[Y^2+2B_1Y+R_b^2] \qquad (3.4)
\end{aligned}
$$

with unchanged definitions of A_1 to R_b^2 from (2.2). We have therefore shown

COROLLARY 3.1. *Theorem 2.1 remains valid if among the pairs $\{a, \bar{a}\}$, $\{b, \bar{b}\}$ are pairs of real distinct points.*

Under the convention that the overbar applied to a real number means another different real number and denotes the complex conjugated point when applied to a non-real point, Theorem 2.1 together with Corollary 3.1 covers the whole class under consideration. The reader now understands why (2.2) was written in that apparently strange form. The deferred question can be raised as follows: Are there points $(a, b, B, n) \in \mathbf{S}_0$ which allow for solutions X of $L[X]=0$ which are not asymptotically stable?

It is not hard to see that for small and large $|B|$, $\mathbf{C}_T$ consists of two compact components around a, $\bar{a}$ (small $|B|$) or b, $\bar{b}$ (large $|B|$) and a non-compact component. For each $|B|$, the vertical line $\mathrm{Re}(z)=x_0:=\ln(|B|)/T$ is the asymptote on the non-compact component of $\mathbf{C}_T$. It follows that $\mathrm{Re}(z)\geqslant 0$ contains for $|B|>1$ infinitely many zeros. For $|B|<1$, $\mathrm{Re}(z)\geqslant 0$ contains only finitely many zeros. Both arguments apply also for the general case of (3.3); $\deg(A)=\deg(B)$, A, B are real polynomials. For $|B|=1$ there are in our case only finitely many zeros for $\mathrm{Re}(a-b)<0$ in $\mathrm{Re}(z)\geqslant 0$. For $\mathrm{Re}(b-a)\geqslant 0$, there are infinitely many there. Application of this argument to Theorem 3.1 gives

COROLLARY 3.2. *Asymptotic stability in the sense of Liapunov is guaranteed whenever $M(a, b)\leqslant 1-\varepsilon$ for arbitrarily small $\varepsilon>0$.*

A fact worth mentioning is the exponential decay of the stability boundary $M^n(a, b)$ with respect to the order n of the zeros of the coefficient polynomials $A(z)$, $B(z)$ in our case. Clearly, this is also for the general case with all zeros of A, B of the same multiplicity true. Have in mind that $\mathbf{S}_0$ is an open manifold. View $H(z)$ for $B=0$ as the original system which is to be stabilized by a control derived from the delayed state $X(t-T)$. Given the real monic $A(z)$ in (3.3) with zeros in $\mathrm{Re}(z)<0$, then choose a real monic $B(z)$ of the same degree as $A(z)$ so that $\mathbf{S}_0$ becomes as large as possible. The answer is $B(z):=A(-z)$. In other words, $A(-z)$ is the optimal delay-independent stabilizer for $A(z)$. To see this, verify that the resulting $H(z)$ has no zeros in $\operatorname{Re}(z) \geqslant 0$ for $|B| < 1$; all its zeros lie on $\operatorname{Re}(z) = 0$ for $|B| = 1$ and all zeros fall in $\operatorname{Re}(z) > 1$ for $|B| > 1$.

In some regions of **P**, the delay-dependent stability of $H(z)$ from (1.2) is qualitatively comparable with that of $H(z) := [(z-a)(z-\bar{a})]^n + B\exp(-zT)$. A detailed comparison is, however, beyond the scope of this article. The interested reader is referred for the last $H(z)$ to Cooke and Grossman [9] or the present author [1]. The delay-dependent stability analysis for the simpler case $H(z) := (z-a)^n + B \cdot (z-b)^n \exp(-zT)$, $a, b \in \mathbf{R}$, can be found in [4]. For the subcase $n = 1$ thereof, see also Liu [15]. In order to treat characteristic equations $H(z)$ belonging to equations with several discrete delays or equations with distributed delays, Cooke and van den Driessche [10] considered

$$H(z) := A(z) + B(z)\exp(-zT) \tag{3.5}$$

with analytical coefficients $A(z)$, $B(z)$. F. Brauer [7, Theorem 1] proved a theorem in which conditions for lag-independent stability for $H(z)$ from (3.5) are given. The conditions of [10, 7] do not allow the treatment of the neutral case. We shall give a version of Brauer's theorem which encompasses the neutral case.

THEOREM 3.3. *Let* $A(z)$, $B(z)$ *in* (3.5) *be holomorphic in an open set containing* $\operatorname{Re}(z) \geqslant 0$. *If* $A(z) \neq 0$ *in* $\operatorname{Re}(z) \geqslant 0$ *and*

$$\begin{gathered} |A(z)| > |B(z)| \qquad \text{on} \quad \operatorname{Re}(z) = 0, \\ \limsup\{|B(z)/A(z)|\colon |z| \to \infty,\ \operatorname{Re}(z) \geqslant 0\} \leqslant L < 1 \end{gathered} \tag{3.6}$$

then $H(z)$ *from* (3.5) *is zero-free in* $\operatorname{Re}(z) \geqslant 0$ *for all* $T \geqslant 0$.

Proof. Denote by $\mathbf{D} := \{z \in \mathbf{C}\colon \operatorname{Re}(z) \geqslant 0,\ |z| \leqslant R\}$ the right semi-disk with radius $R \geqslant 0$ centred about the origin. Let $\partial\mathbf{D} := \partial\mathbf{D}_1 \cup \partial\mathbf{D}_2$ be the boundary of $\mathbf{D}$ where $\partial\mathbf{D}_1$ is the closed interval of $\partial\mathbf{D}$ on $\operatorname{Re}(z) = 0$ and $\partial\mathbf{D}_2$ is the closed circular arc of $\partial\mathbf{D}$. We have to show that $H(z) \neq 0$ in $\mathbf{D}$ for all $R \geqslant 0$. Given $L < 1$, we can choose $\varepsilon > 0$ so that $L + \varepsilon < 1$. Then, by definition of L,

$$|B(z)/A(z)| \leqslant L + \varepsilon \qquad \text{for} \quad R \geqslant R(\varepsilon) \tag{3.7}$$

when $R(\varepsilon)$ is chosen sufficiently large. Consider the functions $F(z) := F_1(z) + F_2(z)$, $F_2(z) := B(z)/A(z)$, $F_1(z) := \exp(zT)$ in $\mathbf{D}$ for $R \geqslant R(\varepsilon)$. The zeros of $H(z)$ are those of $F(z)$. Now, $F(z)$ and $F_2(z)$ are holomorphic in $\operatorname{Re}(z) \geqslant 0$ because $A(z) \neq 0$ in $\operatorname{Re}(z) \geqslant 0$. Obviously,

$$\min\{|F_1(z)|\colon z \in \partial\mathbf{D}\} = 1. \tag{3.8}$$

From (3.7) follows

$$\max\{|F_2(z)|: z \in \partial\mathbf{D}_2\} < 1. \tag{3.9}$$

The first condition of (3.6) implies

$$\max\{|F_2(z)|: z \in \partial\mathbf{D}_1\} < 1. \tag{3.10}$$

Combining (3.9) and (3.10) yields

$$\max\{|F_2(z)|: z \in \partial\mathbf{D}\} < 1. \tag{3.11}$$

In other words, $F_1(z)$, $F_2(z)$ are holomorphic in $\mathbf{D}$ and fulfill $|F_1(z)| > |F_2(z)|$ on $\partial\mathbf{D}$. Rouché's Theorem tells us that $F(z)$ has as many zeros in $\mathbf{D}$ as $F_1(z)$ has in $\mathbf{D}$, i.e., none. Since $R \geqslant R(\varepsilon)$ can be made arbitrarily large, $H(z) \neq 0$ in $\operatorname{Re}(z) \geqslant 0$. ∎

A simple example may demonstrate that there are $H(z)$ which are not accessible when "limsup" in (3.6) is replaced by "lim." For $H(z)$ from (1.2) this replacement changes nothing. In

$$H(z) := a + \exp(z) + B \cdot \{\exp[-z(T-1)] + \exp[-z(T+1)]\}/2,$$
$$|B| < 1 - |a|, \ |a| < 1, \tag{3.12}$$

we have $A(z) := a + \exp(z)$, $B(z) := B \cdot \cosh(z)$, and

$$F_2(z) := B(z)/A(z) = B \cdot \cosh(z)/[a + \exp(z)]. \tag{3.13}$$

$H(z)$ from (3.12), being for simplicity a characteristic equation for a pure difference equation, is stable for all $T \geqslant 0$ and $F_2(z)$, taken along $\operatorname{Re}(z) = 0$, shows that $\lim\{|F_2(z)|: |z| \to \infty, \operatorname{Re}(z) \geqslant 0\}$ does not exist.

There is a minute difference in the idea behind the proofs of Theorems 2.1 and 3.3. In the latter, conditions are sought under which $H(z) \neq 0$ in $\operatorname{Re}(z) \geqslant 0$ resulting in a condition on $\operatorname{Re}(z) = 0$ and a second one "at infinity" in $\operatorname{Re}(z) \geqslant 0$. In the case of Theorem 2.1, conditions which confine all zeros of $H(z)$ in $\operatorname{Re}(z) < 0$ are looked for. The second point of view, when applied to less simple $H(z)$, say $H(z)$ from (3.3), leads to the following strategy for obtaining sufficient stability conditions. Let $\mathbf{L}$ be the (too complicated) solution set of

$$|R(z)| < |B|, \qquad R(z) := A(z)/B(z). \tag{3.14}$$

Then construct a simplified $R_*(z)$ so that $|R_*(z)| < |R(z)|$ holds and the solution set L_* of

$$|R_*(z)| < |B| \tag{3.15}$$

can be computed effectively. Clearly, $\mathbf{L} \subseteq \mathbf{L}_*$. Assume further that $R_*(z)$ has been chosen so that the set $\mathbf{S}_*$ in the parameter space belonging to the $H(z)$ under consideration for which $\mathbf{L}_*$ lies entirely in $R(z) < 0$ can be obtained explicitly. Then $H(z)$ is (at least) stable in $\mathbf{S}_*$ for all $T \geqslant 0$. Such a sufficient lag-independent stability condition for $H(z)$ from (1.2) can be found as follows. We infer from (2.5)

$$\begin{aligned}
&\min\{|(z-a)/(z-b)|^2, |(z-\bar{a})/(z-\bar{b})|^2\} \\
&\qquad \leqslant |(z-a)(z-\bar{a})/[(z-b)(z-\bar{b})]| \leqslant |B|^{1/n}, \\
&\min\{|(z-a)/(z-b)|, |(z-\bar{a})/(z-\bar{b})|\} \\
&\qquad \leqslant |B|^{1/(2n)}.
\end{aligned} \tag{3.16}$$

The solution set $\mathbf{L}_*$ of the last inequality is the union of two complex conjugated disks and $\mathbf{S}_*$ specifies the pairs (a, b) for which the disks lie entirely in $\mathrm{Re}(z) < 0$. This leads to a stability condition of the form

$$|B| < M_*^{2n}(a, b). \tag{3.17}$$

For more details on $M_*(a, b) > 0$, see [5, Sect. 4].

Brauer's examples for the polynomial case, besides other cases, may also be found in the author's papers [1, 2] which appeared earlier or at about the same time. For the intricacies of the neutral case, see the papers of Gromova cited in [11] and Brumley's paper [8] besides the two classical papers of Hahn [12, 14] along with his remarks on [12] in [13]. A less explicit version of Theorem 2.1 can also be found in [6]. The calculation of the delay-dependent stability set in parameter space is normally much more complicated than the delay-independent set (see [3] or [9] for examples).

References

1. F. G. Boese, On the stability of the linear difference–differential equation with constant coefficients and one constant delay, *Z. Angew. Math. Mech.* **66** (1986) 4, T27–T29.
2. F. G. Boese, Some stability charts and stability conditions for a class of difference-differential equations, *Z. Angew. Math. Mech.* **67** (1987) 4, T56–T59.
3. F. G. Boese, The stability chart for the linearized Cushing equation with a discrete delay and with gamma-distributed delays, *J. Math. Anal. Appl.* **140** (1989), 510–536.
4. F. G. Boese, The stability chart for a sequence of linear neutral difference–differential equations with one delay, *Z. Angew. Math. Mech.* **70** (1990) 6, T607–T610.
5. F. G. Boese, Stability conditions for the general linear difference–differential equation with constant coefficients and one constant delay, *J. Math. Anal. Appl.* **140** (1989), 136–176.
6. F. G. Boese, "Zur Lage der Nullstellen von Exponentialpolynomen und zur Stabilität von Exponentialpolynomen," Habilitationsschrift, Universität Hamburg, Hamburg, 1989.

7. F. BRAUER, Absolute stability in delay equations, *J. Differential Equations* **69** (1987), 185–191.
8. W. E. BRUMLEY, On the asymptotic behavior of solutions of differential–difference equations of neutral type, *J. Differential Equations* **7** (1970), 175–188.
9. K. L. COOKE AND Z. GROSSMAN, Discrete delays, distributed delays and stability switches, *J. Math. Anal. Appl.* **66** (1982), 592–627.
10. K. L. COOKE AND P. VAN DEN DRIESSCHE, On zeroes of some transcendental functions, *Funkcial. Ekvac.* **29** (1986), 77–90.
11. L. E. EL'SGOL'TS AND S. B. NORKIN, "Introduction to the Theory and Application of Differential Equations with Deviating Arguments," Academic Press, New York, 1973.
12. W. HAHN, Zur Stabilität der Lösungen von linearen Differential–Differenzengleichungen, *Math. Ann.* **131** (1956), 151–166.
13. W. HAHN, Bemerkungen zur Arbeit über Differential–Differenzgleichungen, *Math. Ann.* **132** (1956), 94.
14. W. HAHN, Über Differential–Differenzengleichungen mit anomalen Lösungen, *Math. Ann.* **133** (1957), 251–255.
15. CHANGMEI LIU, The neutral-type equation $\dot{x}(t)+a\dot{x}(t-\tau)+bx(t)+cx(t-\tau)=0$, *J. Hai Nan Univ. Science Edition* **3** (1985) 4, 60–70. [in Chinese]